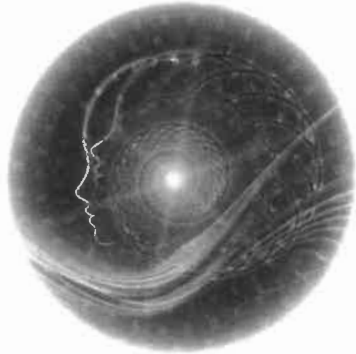
JO BOALER

FOREWORD BY CAROL DWECK

MATHEMATICAL MINDSETS



Unleashing Students' POTENTIAL Through Creative Math, Inspiring Messages and INNOVATIVE TEACHING

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CHAPTER 5

Rich Mathematical Tasks

Teachers are the most important resource for students. They are the ones who can create exciting mathematics environments, give students the positive messages they need, and take any math task and make it one that piques students' curiosity and interest. Studies have shown that the teacher has a greater impact on student learning than any other variable (Darling-Hammond, 2000). But there is another critical part of the mathematics learning experience—in many ways, it is a teacher's best friend—and that is the curriculum teachers get to work with, the tasks and questions through which students learn mathematics. All teachers know that great mathematics tasks are a wonderful resource. They can make the difference between happy, inspired students and disengaged, unmotivated students. The tasks and questions used help develop mathematical mindsets and create the conditions for deep, connected understanding. This chapter will delve into the nature of true mathematics engagement and consider how it is brought about through the design of mathematics tasks.

I have taught mathematics at middle school, high school, and undergraduate levels in England and the United States. I have also observed and researched hundreds of mathematics classrooms, across the K–16 level, in both countries, and studied students' learning of mathematics and the conditions that bring it about. I am fortunate to have had such a broad experience for many reasons, one of them being it has given me a great deal of insight into the nature of true mathematics engagement and deep learning. I have witnessed mathematical excitement, as it happens, with a range of different students, leading to the development of precious insights into mathematical ideas and relationships. Interestingly, I found that mathematics excitement looks exactly the same for struggling 11-year-olds as it does for high-flying students in top universities—it combines *curiosity*, *connection making*, *challenge*, and *creativity*, and usually involves *collaboration*. These, for me, are the 5 C's of mathematics engagement. In this chapter I will share what I have learned about the nature of mathematics engagement and excitement before considering the qualities of tasks that produce such engagement—tasks that all teachers can create in their own mathematics classrooms.

Rather than dissecting the nature of mathematics engagement in a clinical and abstract way, I want to introduce you to five cases of true mathematics excitement. I think of mathematics excitement as the pinnacle of mathematics engagement. These are all cases that I have personally witnessed among groups of people and that have given me important insights into the nature of the teaching and tasks that bring about such learning opportunities. The first case comes not from a school but from the unusual setting of a startup company in Silicon Valley. This case shows something powerful about mathematical excitement that I would love to capture and bottle for all teachers of mathematics.

Case 1. Seeing the Openness of Numbers

It was late December 2012, days before I was to fly to London for the holidays, when I first met Sebastian Thrun and his team at Udacity, a company producing online courses. I had been asked to visit Udacity to give the team advice about mathematics courses and ways to design effective learning opportunities. I walked into the airy space in Palo Alto that day and knew immediately that I had walked into a Silicon Valley startup—bikes were suspended on walls; young people, mostly men, wearing T-shirts and jeans, pored over computers or sat chatting about ideas; there were no office walls, only partitioned cubicles and light. I walked through the cubicles to the conference room at the back behind a glass wall. About 15 people had squeezed into the small room, sitting on chairs and the floor. Sebastian stepped forward and shook my hand, made some introductions, and invited me to sit down. He started firing questions at me: "What makes a good math course? How should we teach math? Why are students failing math?" He said that his friend Bill Gates had told him algebra was the reason we have widespread math failure in the United States. I cheekily replied, "Oh, Bill Gates the math educator told you that, did he?" His team members smiled, and Sebastian looked momentarily taken aback. He then asked, "Well, what do *you* think?" I told him that students were failing algebra not because algebra is so difficult, but because students don't have number sense, which is the foundation for algebra. Chris, one of the course designers who was also a former math teacher, nodded in agreement.

Sebastian continued firing questions at me. When he asked me what makes a good math question, I stopped the conversation and asked the group if I could ask *them* all a math question. They readily agreed, and I enacted a mini version of a number talk. I asked everyone to think about the answer to 18×5 and to show me, with a silent thumbs-up, when they had an answer. The thumbs started to pop up, and the team shared methods. There were at least six different methods shared that day, which I drew, visually, on the write-on table we sat around (see Figure 5.1).

We then discussed the ways the different methods were similar and different. As I drew the visual methods, the team members' eyes grew wider and wider. Some of them started to hop in their seats with excitement. Some shared that they had never imagined that there were so many ways to think about an abstract number problem; others said they were amazed that there was a visual image and it showed so much of the mathematics, so clearly.

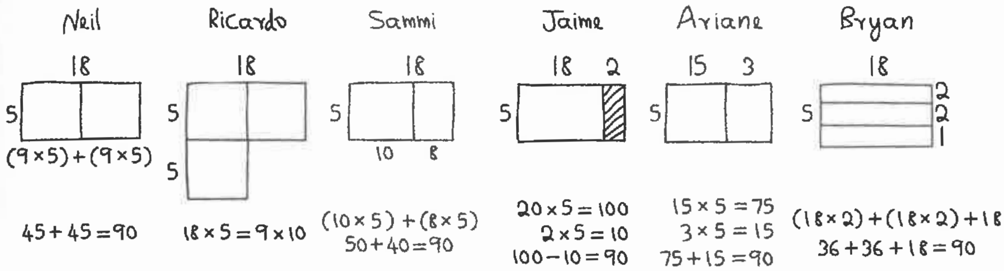


FIGURE 5.1 Visual solutions to 18 × 5

When I arrived in London, a few days later, I opened an email from Andy, the innovative young course designer at Udacity. He had made a mini online course on 18 × 5, which included going out on the street and interviewing passersby, collecting different methods. The team had been so excited by the ideas that they wanted to immediately put them out to the public, and they talked about making 18 × 5 T-shirts for everyone at Udacity to wear.

In the months following the meeting at Udacity, I met Luc Barthelet, then a director of Wolfram Alpha, one of the most important mathematics companies in the world. Luc had read about the different solutions to 18 × 5 I had shown in my book (Boaler, 2015a), which so excited him that he started asking everyone he met to solve 18 × 5. These reactions, these moments of intense mathematics excitement around an abstract number problem, seem important to share. How is it possible that these high-level math users, as well as young children, are so engaged by seeing and thinking about the different methods people use to solve a seemingly unexciting problem like 18 × 5? I propose that this engagement comes from people seeing the creativity in math and the different ways people *see* mathematical ideas. This is intrinsically interesting, but it's also true that most people I meet, even high-level mathematics users, have never realized numbers can be so open and number problems can be solved in so many ways. When this realization is combined with visual insights into the mathematical ways of working, engagement is intensified.

I have used this and similar problems with middle school students, Stanford undergrads, and CEOs of companies, all with equal engagement. I have learned through this that people are fascinated by flexibility and openness in mathematics. Mathematics is a subject that allows for precise thinking, but when that precise thinking is combined with creativity, flexibility, and multiplicity of ideas, the mathematics comes alive for people. Teachers can create such mathematical excitement in classrooms, with any task, by asking students for the different ways they see and can solve tasks and by encouraging discussion of different ways of seeing problems. In classrooms, teachers have to pay attention to classroom norms and teach students to listen to and respect each other's thinking; Chapter Seven will show a teaching strategy for this. When students have learned norms of respect and listening, it is incredible to see their engagement when different ways to solve a problem are shared.

Case 2. Growing Shapes: The Power of Visualization

The next case I want to share comes from a very different setting—a middle school classroom in a San Francisco Bay Area summer school where students had been referred because they were not performing well in the school year. I was teaching one of the four math classes with my graduate students at Stanford. We had decided to focus the classes on algebra, but not algebra as an end point, with students mindlessly solving for x. Instead, we taught algebra as a problem-solving tool that could be used to solve rich, engaging problems. The students had just finished sixth and seventh grades, and most of them hated math. Approximately half the students had received a D or an F in their previous school year (for more detail, see Boaler, 2015a; Boaler & Sengupta-Irving, 2015).

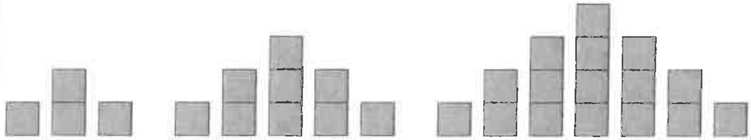
In developing a curriculum for the summer school, we drew upon a range of resources, including Mark Driscoll's books, Ruth Parker's mathematics problems, and two curricula from England—SMILE (which stands for secondary mathematics individualized learning experience) and Points of Departure. The task that created this case of mathematics excitement came from Ruth Parker; it asked the students to extend the growing pattern shown in Exhibit 5.1, made out of multilink cubes, to find how many cubes there would be in the 100th case. (Full-page task worksheets of all exhibits can be found in the Appendix.)

The students had multilink cubes to work with. In our teaching we invited the students to work together in groups to discuss ideas, sometimes groups we teachers chose, other times groups the students chose. On the day in question, I noticed an interesting grouping of three boys—three of the naughtiest boys in my class! They did not know each other before coming to the summer school, but all three spent most of the first week of summer school either off task or working to pull others off task. The boys would shout things out when others were at the board showing math and generally seemed more interested in social conversations than math conversations in the early days. Jorge had received an F in his last math class, Carlos a D, and Luke an A. But the day we gave the students this task, something changed. Incredibly, the three boys worked on this math task for 70 minutes, without ever stopping, becoming distracted, or moving off task. At one point some girls came over and poked them with pencils, which caused the boys to pick up their work and move to another table, they were so intensely engaged in the task and working to solve the problem.

All of our lessons were videotaped, and when we reviewed the film of the boys working that day we watched a rich conversation about number patterns, visual growth, and algebraic generalization. Part of the reason for the boys' intense engagement was an adaptation to the task that we had used—an adaptation that can be used with any math task. In classrooms, typically when function tasks such as the one we gave to the students are assigned, they are usually given with the instruction to find the 100th case (or some other high number) and ultimately the nth case. We did not start with this. Instead, we asked the students to first think alone, before moving to group work, about the ways they *saw* the shape growing. We encouraged them to think visually, not with numbers, and to sketch in their journals, showing us where they saw the extra cubes in each case. The boys saw the growth of the shape in different ways. Luke and Jorge saw the growth as cubes added to the bottom of the shape each time; this later became known by the class as

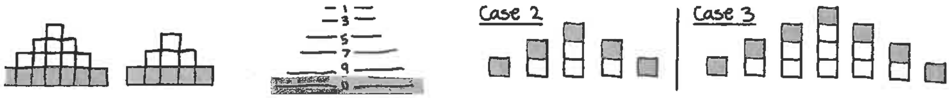
Shapes Task

How do you see the shapes growing?



Source: From Ruth Parker; a task used in MEC courses.

Exhibit 5.1



Luke Jorge Carlos

ɪURE 5.2 Students' work
rce: Selling, 2015.

"bowling alley method," as the cubes arrived like a new line of pins in a bowling alley. Carlos the growth as cubes added to the top of the columns—what became known as the "raindrop :hod"—cubes dropping down from the sky, like raindrops, onto the columns (see Figure 5.2). .fter the boys had spent time working out the function growth individually, they shared with ı other their ideas for how the shape was growing, talking about where they saw the additional es in each case. Impressively, they connected their visual methods with the numbers in the pes, not only working with their own methods but taking the time to explain the different :hods to each other and using each other's methods. The three boys were intrigued by the ction growth and worked hard to think about the 100th case, armed with their knowledge he visual growth of the shape. They proposed ideas to each other, leaning across the table and nting to their journal sketches. As is typical for mathematical problem solving, they zigzagged und, moving close to a solution, then further away, then back toward it again (Lakatos, 1976). :y tried different pathways to the solution, and they broadly explored the mathematical terrain. have shown a video of the boys working to many conference audiences of teachers, and all e been highly impressed with the boys' motivation, perseverance, and high-level mathematical ıversation. Teachers know that the perseverance shown by the three boys and the respectful 's they discussed each others' ideas, particularly in the context of summer school, is highly ısual, and they are curious as to how we were able to bring it about. They know that many stu-ıts, particularly those who have been unsuccessful, give up when a task is hard and they don't nediately know the answer. That didn't happen in this case; when the boys were stuck, they looked back at their diagrams and tried out ideas with each other, many of which were incorrect but helpful in ultimately forming a pathway to the solution. After watching the case with teachers at conferences, I ask them what they see in the boys' interactions that could help us understand their high level of perseverance and engagement. Here are some important observations that reveal opportunities to improve the engagement of all students:

1) **The task is challenging but accessible.** All three boys could access the task, but it provided a challenge for them. It was at the perfect level for their thinking. It is very hard to find tasks that are perfect for all students, but when we open tasks and make them broader—when we make them what I refer to as "low floor, high ceiling"—this becomes possible for all students. The floor is low because anyone can see how the shape is growing, but the ceiling is high—the function the boys were exploring is a quadratic function whereby case n can be represented by $(n+1)^2$ blocks. We made the floor of the task lower by inviting the students to think visually about the case—although, as I will discuss later, this was not the only reason for this important adaptation.

2) **The boys saw the task as a puzzle,** they were curious about the solution, and they wanted to solve it. The question was not "real world" or about their lives, but it completely engaged them. This is the power of abstract mathematics when it involves open thinking and connection making.

3) **The visual thinking about the growth of the task gave the boys understanding of the way the pattern grew.** The boys could see that the task grew as a square of $(n+1)$ side length because of their visual exploration of the pattern growth. They were working to find a complex solution, but they were confident in doing so, as they had been given visual understanding to help them.

4) The boys were encouraged by the fact that **they had all developed their own way of seeing the pattern growth** and their different methods were valid and added different insights into the solution. The boys were excited to share their thinking with each other and use their own and each other's ideas in the solving of the problem.

5) **The classroom had been set up to encourage students to propose ideas without being afraid of making mistakes.** This enabled the boys to keep going when they were "stuck," by providing ideas, right or wrong, that enabled the conversation to continue.

6) We had taught the students to **respect each other's thinking.** We did this by valuing the breadth of thinking everyone could offer, not just the procedural thinking that some could offer, valuing the different ways people saw problems and made connections.

7) The students were **using their own ideas,** not just following a method from a book as they learned core algebraic content. The fact that they had proposed different visual ideas for the growth of the function made them more invested and interested in the task.

8) **The boys were working together**; the video shows clearly the way the boys built understanding through the different ideas they shared in conversation, which also enhanced their enjoyment of the mathematics.

9) The boys were working heterogeneously. Viewers of the video note that each boy offers something different and important. The high achiever keeps shouting out number guesses—something that may have been a successful strategy with more procedural questions—but the lower-achieving boys push him to think visually and ultimately more conceptually, and it is the combination of the different boys' thinking that ultimately helps them and leads to success.

Typically, growth pattern tasks are given to students with a numerical question such as "How many cubes are in the 100th case?" and "How many cubes are in the nth case?" We also asked students these questions, but we prefaced them with individual time in which students considered the visual growth of the shape. That changed everything. People think about the growth of the shape in many different ways, as shown in Figures 5.3 through 5.10. When we don't ask students to think visually, we miss an incredible opportunity to increase their understanding. These are some of the ways teachers and students I have worked with see the growth of the shape, accompanied by names they use to capture the growth.

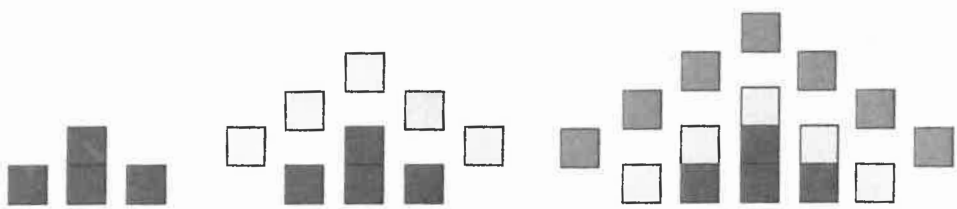


FIGURE 5.3 The Raindrop Method—cubes come from the sky like raindrops

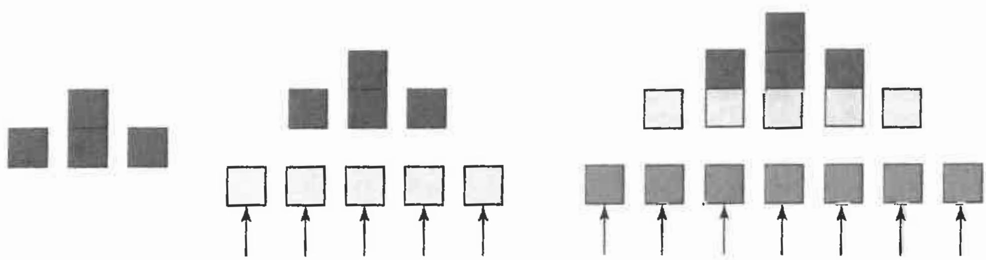


FIGURE 5.4 The Bowling Alley Method—cubes are added like pins in a bowling alley

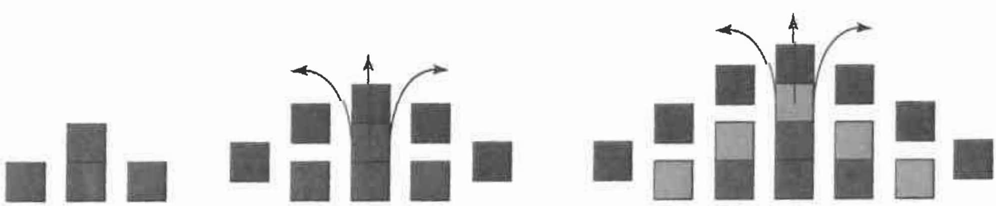


FIGURE 5.5 The Volcano Method—the middle column of cubes grows high and the rest follow like lava erupting from a volcano

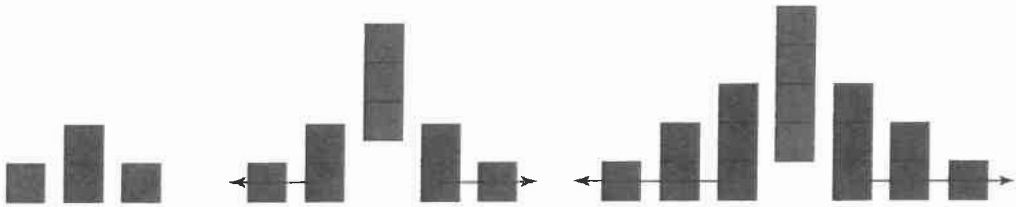


FIGURE 5.6 The Parting of the Red Sea Method—the columns part and the middle column arrives

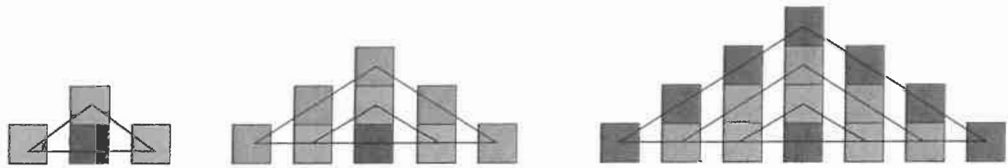


FIGURE 5.7 The Similar Triangles Method—the layers can be seen as triangles

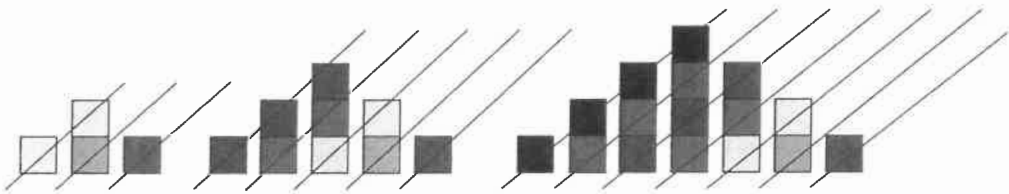


FIGURE 5.8 The Slicing Method—the layers can be viewed diagonally

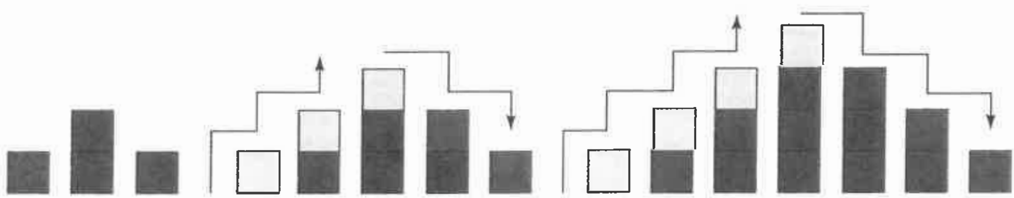


FIGURE 5.9 "Stairway to Heaven, Access Denied"—from *Wayne's World*

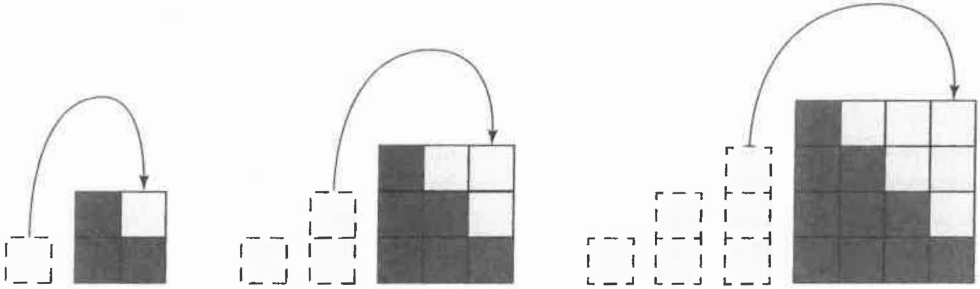


FIGURE 5.10 The Square Method—the shapes can be rearranged as a square each time

I recently gave this pattern-growth task to a group of high school teachers who did not take the time to explore the visual growth of the shape and instead produced a table of values like this:

case	#cubes
1	4
2	9
3	16
n	$(n+1)^2$

When I asked the teachers to tell me why the function was growing as a square, why it was (n+1) *squared*, they had no idea. But this is why we see a squared function: the shape grows as a square, with a side of (n+1), where n is the case number (see Figure 5.11).

When we do not ask students to think visually about the growth of the shape, they do not have access to important understanding about functional growth. They often cannot say what "n" means or represents, and algebra remains a mystery to them—a set of abstract letters they

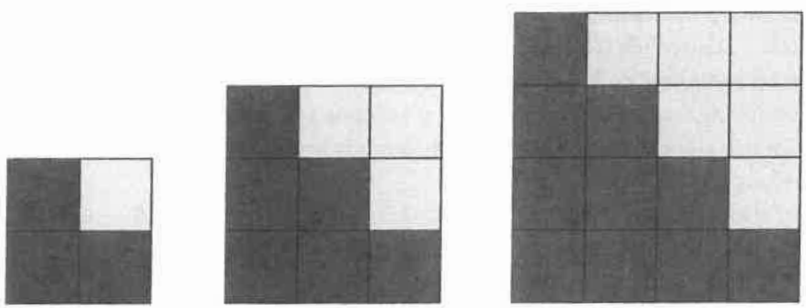


FIGURE 5.11 The Square Method 2

move around on a page. Our summer school students knew what "n" represented, because they had drawn it for themselves. They knew why the function grew as a square and why the nth case was represented by $(n+1)^2$. The algebraic expression they ultimately produced was meaningful to them. Additionally, students did not think they were finding a standard answer for us; they thought they were exploring methods and using their own ideas and thoughts, which included their own ways of seeing mathematical growth. In the final section of this chapter, I will review the ways these features of this task can be used in other tasks to produce increased student engagement and understanding.

Case 3. A Time to Tell?

When I share open, inquiry-based mathematics tasks with teachers, such as the growing shapes or "raindrop" task just discussed, they often ask questions such as, "I get that these tasks are engaging and create good mathematical discussions, but how do students learn new knowledge, such as trig functions? Or how to factorize? They can't discover it." This is a reasonable question, and we do have important research knowledge about this issue. It is true that while ideal mathematics discussions are those in which students use mathematical methods and ideas to solve problems, there are times when teachers needs to introduce students to new methods and ideas. In the vast majority of mathematics classrooms, this happens in a routine of teachers showing methods to students, which students then practice through textbook questions. In better mathematics classrooms, students go beyond practicing isolated methods and use them to solve applied problems, but the order remains—teachers show methods, then students use them.

In an important study, researchers compared three ways of teaching mathematics (Schwartz & Bransford, 1998). The first was the method used across the United States: the teacher showed methods, the students then solved problems with them. In the second, the students were left to discover methods through exploration. The third was a reversal of the typical sequence: the students were first given applied problems to work on, even before they knew how to solve them; then they were shown methods. It was this third group of students who performed at significantly higher levels compared to the other two groups. The researchers found that when students were given problems to solve, and they did not know methods to solve them, but they were given opportunity to explore the problems, they became curious, and their brains were primed to learn new methods, so that when teachers taught the methods, students paid greater attention to them and were more motivated to learn them. The researchers published their results with the title "A Time for Telling," and they argued that the question is not "Should we *tell* or explain methods?" but "When is the best time do this?" Their study showed clearly that the best time was *after* students had explored the problems.

How does this work in a classroom? How do teachers give students problems that they cannot solve without the students experiencing frustration? In describing how this works in practice, I will draw from two different cases of teaching.

The first example comes from the research study I conducted in England that showed that students who learned mathematics through a project-based approach achieved at significantly higher levels in mathematics, both in standardized tests (Boaler, 1998) and later in life (Boaler, 2005), than students who worked traditionally. In one of the tasks I observed in the project school, a group of 13-year-old students were told that a farmer wanted to make the largest enclosure she could out of 36 1-meter pieces of fencing. The students set about investigating ways to find the maximum area. Students tried different shapes, such as squares, rectangles, and triangles, and tried to find a shape with the biggest possible area. Two students realized that the biggest area would come from a 36-sided shape, and they set out to determine the exact area (see Figure 5.12).

They had divided their shape into 36 triangles, and they knew the base of each triangle was 1 meter and the angle at the vertex was 10 degrees (see Figure 5.13).

However, this alone was not enough to find the area of the triangle. At this point the teacher of the class showed the students trigonometry and the ways that a sine function could be used to give them the height of the triangle. The students were thrilled to learn this method, as it helped them solve the problem. I watched as one boy excitedly taught his group members how to use a sine function, telling them he had learned something "really cool" from the teacher. I then remembered the contrasting lesson I had watched in the traditional school a week earlier, in which the teacher had given the students trig functions and then pages of questions to practice them on. In that case, the students had thought the trig functions were extremely boring and unrelated to their lives. In the project school, the students were excited to learn about

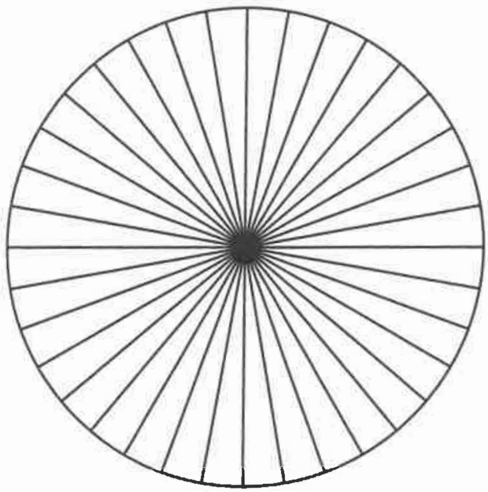


FIGURE 5.12 A 36-sided fence yields the largest enclosure area

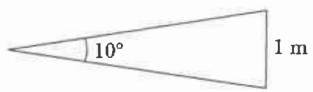


FIGURE 5.13 The triangle formed from a 1-meter fence section

trigonometry and saw the methods as "cool" and useful. This heightened motivation means that the students learned the methods more deeply, and it is a large part of the success of the students in the project school in examinations and life.

The second example of students learning methods after they were given problems came from a research study I conducted in the United States. Similar to the UK study, it showed that students learned at significantly higher levels when taught mathematics through a conceptual approach focused on connections and communications (Boaler & Staples, 2005). More detail on both school approaches is given in my book *What's Math Got to Do with It?* (Boaler, 2015a). One day I was in one of the pre-calculus classrooms of the successful school, which I called Railside, when the teacher taught a lesson focused on finding the volume of a complex shape. The teacher, Laura Evans, was preparing the students to learn calculus and to find an area under a curve using integrals, but she did not, as typically happens, teach the formal method to the students first. Instead, she gave students a problem that needed this knowledge and asked them to think about how they would solve it. The problem was to work out a way to find the volume of a lemon. To think about this, she gave each group a lemon and a large knife and asked them to explore possible solutions (see Figure 5.14).

FIGURE 5.14 What is the volume of a lemon?
Source: Shutterstock, Copyright: ampFotoStudio.

After groups had discussed the problem, different students came to the board to excitedly share their ideas. One group had decided to plunge the lemon into a bowl of water to measure the displacement of the water. Another had decided to carefully measure the size of the lemon. A third had decided to cut the lemon into thin slices and think of the slices as two-dimensional sections, which they divided into strips, getting close to the formal method for finding the area under a curve that is taught in calculus (see Figure 5.15).

When the teacher explained to students the method of using integrals, they were excited and saw the method as a powerful tool.

In both of these cases the order of teaching methods was reversed. The students learned trig methods and limits *after* exploring a problem and encountering the need for the methods. The teacher taught them the methods when they were needed, rather than the usual approach of teaching a method that students then practiced. This made a world of difference to the students' interest in and subsequent understanding of the methods.

As I recalled in Chapter Four, Sebastian Thrun explained to me the key role played by intuition in his mathematical work. He said that he did not make mathematical progress unless he intuitively felt it was the right direction. Mathematicians also highlight the critical role played by intuition in their work. Leone Burton interviewed 70 research mathematicians and found that 58 of them embraced and talked about the essential role of intuition in their work (Burton, 1999). Hersh draws a similar conclusion when studying mathematical work: "If we look at mathematical

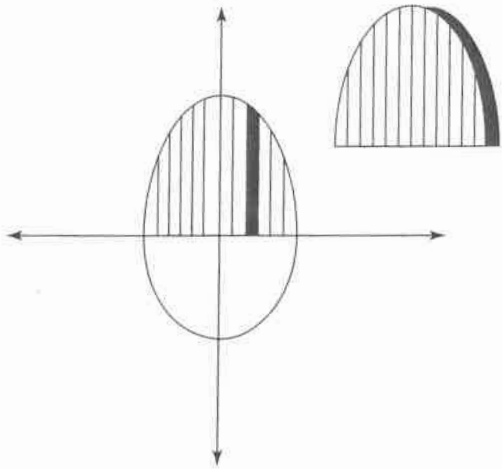


FIGURE 5.15 Calculating a lemon's volume by sections

practice, the intuitive is everywhere" (Hersh, 1999). So why is intuition, so central to mathematics, absent in most math classrooms? Most children do not think intuition is even allowed in their math work. When students were asked to think about finding the volume of a lemon, they were asked to think intuitively about math. Many mathematics problems could be posed to students with the request to think intuitively about ways to solve the problem. Young children could be given different triangles and rectangles and asked to think about ways they might find the area of a triangle, *before* being told an area formula. Students could think about ways to capture the differences between data sets *before* being taught about mean, mode, or range. Students can explore relationships in circles before being told the value of pi. In all cases, when students go on to learn the formal methods, their learning will be deeper and more meaningful. When students are asked to think intuitively, many good things happen. First, they stop thinking narrowly about single methods and consider mathematics more broadly. Second, they realize they have to use their own minds—thinking, sense making, and reasoning. They stop thinking their task is just to repeat methods, and they realize their task is to think about the appropriateness of different methods. And third, as the Schwartz and Bransford research study showed, their brains become primed to learn new methods (Schwartz & Bransford, 1998).

Case 4. Seeing a Mathematical Connection for the First Time (Pascal's Triangle)

My next case of mathematics excitement comes from a professional development workshop I was observing. The teacher of the workshop was Ruth Parker, an amazing educator who offers workshops for teachers that give them access to mathematical understandings they have never had before. I chose this case to share because I saw something that day that I have seen many times since: a task that allowed a teacher, Elizabeth, to see a mathematical connection so powerful it made her cry. The teacher was an elementary teacher who, like many others, had thought of mathematics as a set of procedures to follow. She did not know that mathematics was a subject full of rich connections. It is not uncommon for people who have always believed that mathematics is a disconnected set of procedures to be extremely moved when they see the rich connections that make up mathematics.

Ruth's workshop, similar to our summer school teaching, was focused on algebraic thinking, and she engaged the teachers in many function pattern tasks. The task chosen by Ruth that day was a lovely low floor, high ceiling task that appears simple but leads to wonderful complexity. The Cuisenaire Rod Train Task is shown in Exhibit 5.2. For the teachers in Ruth's workshop, it led to explorations of exponential growth and negative exponents.

Elizabeth and the other teachers set to work on the task, ordering and arranging Cuisenaire rods to find all the ways they could make trains the length of three rods they had chosen. Some teachers in the workshop chose to start with the 10 rod, which made their task extremely difficult—there are 512 ways to make trains that are as long as the 10 rod! Ruth knew that her role as the teacher was not to rescue them, but to let them wade around inside the mathematics of the problem. After a lot of struggle, some of them remembered something they had learned earlier in the workshop—a key mathematical practice students can go through 11 years of school without learning: to try a smaller case. The teachers worked on different rod lengths and started to see a pattern emerging, both visually and numerically (see Exhibit 5.2).

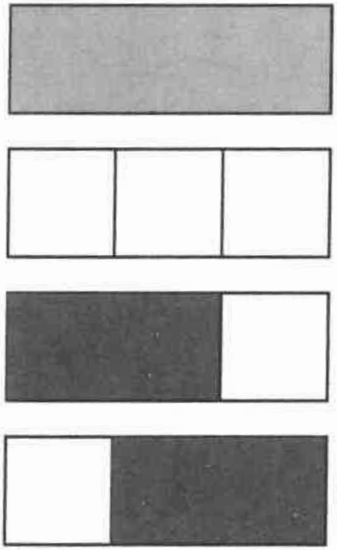
At this point Ruth introduced the teachers to Pascal's triangle, and asked teachers to search for the connection between the Cuisenaire rods problem and the famous triangle (see Exhibit 5.3 and Figure 5.16).

After much struggle the teachers then, with some amazement, saw that their Cuisenaire train combinations were all inside Pascal's triangle. This was the moment when Elizabeth was moved to tears—an emotion I fully understand. For anyone who has only ever seen mathematics as a set of disconnected procedures, who then gets the opportunity to explore visual and numerical patterns, seeing and understanding connections, the experience is powerful. Elizabeth gained an intellectual empowerment in those moments that told her that she could, herself, discover mathematical insights and connections. From that point on, Elizabeth's relationship with math changed, and she never looked back. I caught up with Elizabeth a year later, when she was retaking Ruth's course to experience more powerful mathematics learning, and she told me all of the wonderful ways she had changed her math teaching and the new mathematical excitement that she was seeing in her third-grade students.

Elizabeth's experience of seeing mathematics in an entirely new way when she was introduced to mathematical connections is one I have seen repeated over and over again with different children and adults. The strength of emotions I see relates directly to the experience of seeing, exploring, and understanding mathematical connections.

Cuisenaire Rod Trains

Find out how many different trains can be made for any length of rod. For example, with the light green rod you can make these four trains:



Source: From Ruth Parker; a task used in MEC courses.

Exhibit 5.2

Case 5. The Wonders of Negative Space

This case draws from a task I have used with my teacher education group at Stanford and with different groups of teachers; it creates such intensity of excitement that it seems important to share. The task is again a pattern-growth task but with an added twist; it is that added twist I want to focus on. The pattern-growth task came from Carlos Cabana, a wonderful teacher I work with. Exhibit 5.4 shows the two questions he usually gives to students.

One of the questions posed in the task is how many tiles there would be in Figure −1; in other words, if the pattern was to extend backward to case 1, case 0, and case −1, how many tiles would there be in case −1? In giving this task to teachers, I found that the number of tiles was an easy question for them to answer; what was much more interesting and challenging was the question

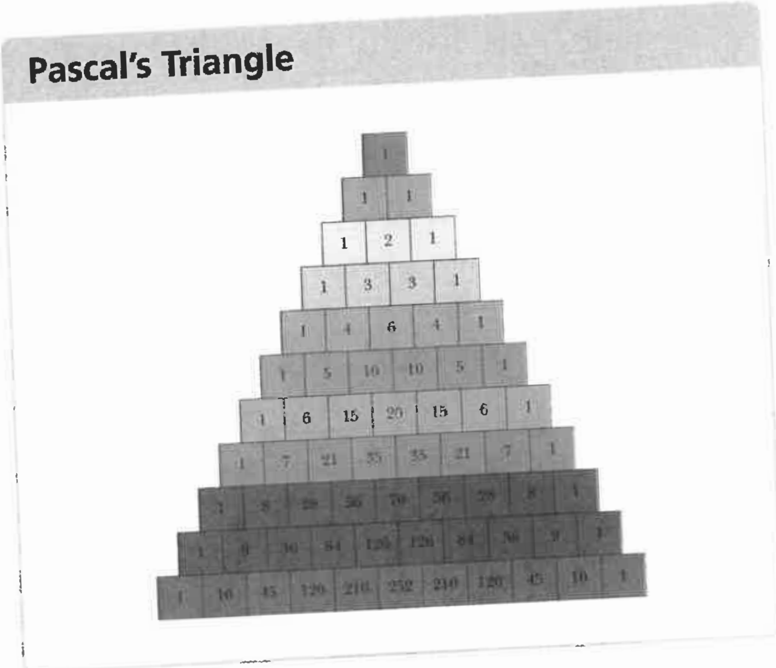


Exhibit 5.3

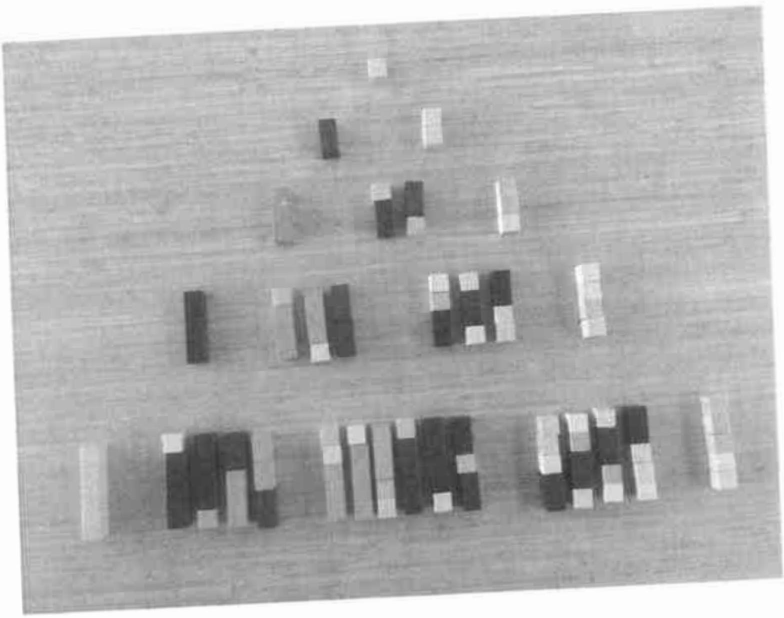


FIGURE 5.16 Pascal's triangle in Cuisenaire rods

Negative Space Task

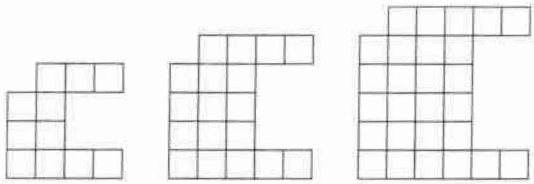


Fig. 2 Fig. 3 Fig. 4

1. What would figure 100 look like?

2. Imagine you could continue your pattern backward. How many tiles would there be in Figure −1? (That's figure negative one, whatever that means!)

3. What would Figure −1 look like?

Source: Adapted from Carlos Cabana.

Exhibit 5.4

of what the negative-one case would look like. When I added this question to the task, amazing things happened. First of all, the solution (which I won't give away) is challenging, and teachers laughed about their heads hurting and synapses firing when trying to figure it out. There is more than one way to get to the negative-one case, and not only is there more than one correct visualization, but there is also more than one numerical solution, because the question moves into unusually uncharted and exciting waters: considering the question of what negative squares look like. Some of the teachers discovered that they would need to think about *negative space,* and what a tile would look like if it was inverted onto itself. When I gave my teacher education group this task at Stanford, they were jumping over tables with excitement trying to represent negative space, poking holes through the paper to show the tiles going into negative space. One of the teachers realized and shared that the function could be represented as a parabola on a graph (see Figure 5.17). Another teacher asked where the parabola would go—would it stay on the positive y axis or flip below the axis?

This question was extremely engaging for the group, and they excitedly tried to work it out. At the end of the session, the teacher candidates reflected that they had now experienced true mathematics excitement and knew what they wanted students in their classes to experience.

But what caused this excitement, which I have now seen repeated in many different places? When I gave this task to a group of teacher leaders in Canada recently, they were so engaged by the task that I could not get them to stop working on it, which they laughed about. People tweeted "Jo Boaler can't bring us back from the task she has set."

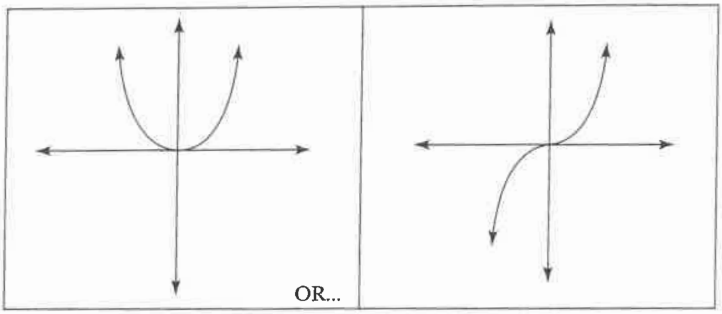


FIGURE 5.17 Parabola Dilemma

One reason the task is so exciting is that it involves thinking about negative space, stepping into another dimension—and that is exciting, period. This is something that mathematics allows us to do, and one reason why mathematics is an exciting subject. Additionally, the students believed that they were exploring uncharted waters; they were not finding an answer to a question that the textbook and teacher knew, and that increased their excitement dramatically. When students were asking about the direction of the parabola, they felt that they could ask anything—that mathematics was an open subject and that when they discovered a new idea (a parabola) they could take it further with another question that they posed. The visual representation of the mathematical pattern was again hugely important for the levels of engagement.

Before reflecting on what these different cases of intense mathematics excitement mean for the design of engaging tasks, I want to introduce one last case, this one from a third-grade classroom.

Case 6. From Math Facts to Math Excitement

Chapter Four discussed the importance of teachers' changing the ways they encourage students to learn math facts, moving from activities that are often traumatic for students—timed tests of isolated facts and hours of memorization—to engaging activities that support important brain connections. To help teachers make such changes, I wrote a paper with my Youcubed colleagues, as I described in the last chapter, entitled "Fluency without Fear," and I posted the paper on our site in the hope that it would reach many teachers, but we could not have anticipated the eventual extent of the impact, with major newspapers across the United States covering the ideas in the paper. One activity that we gave to teachers created a positive impact in a different way, as teachers circulated it widely among themselves using various social media, alongside photos of their students enjoying the activity and making important brain connections.

The activity (described in the preceding chapter) that proved to be so important and popular was a game called "How close to 100?"

One of the teachers who took my online class and subsequently changed her mathematics teaching was RoseAnn Hernandez, a third-grade teacher in a Title 1 school—that is, a school in California where at least 40% of the children are from low-income families. RoseAnn has a copy of Youcubed's seven positive math norms (see Chapter Nine) displayed on her wall for all the students to see. RoseAnn shared with me her students' excitement in playing the game, as well as the important mathematical opportunities that they received (see Figure 5.18). RoseAnn is a very thoughtful teacher; she not only gave the game to students but also prepared them with a discussion before the game; she also prepared extension activities for any students who were working faster. Before the game she asked students to think about the ways dice can be used as a math tool. She gave them opportunities to roll two dice and take turns stating the multiplication factors and products that they generated. She then asked them an important question: How are multiplication and area related? The students thought carefully about this. RoseAnn then invited students to work on the game in pairs and to think about what they were learning as they played. She also challenged the students to decompose the numbers and discover different ways to write the number sentences on the back of their paper if they finished early. The students played the game with much excitement, and when RoseAnn asked them to rate their enjoyment on a 1-to-5 scale, 95% of the students gave it the highest possible rank of 5.

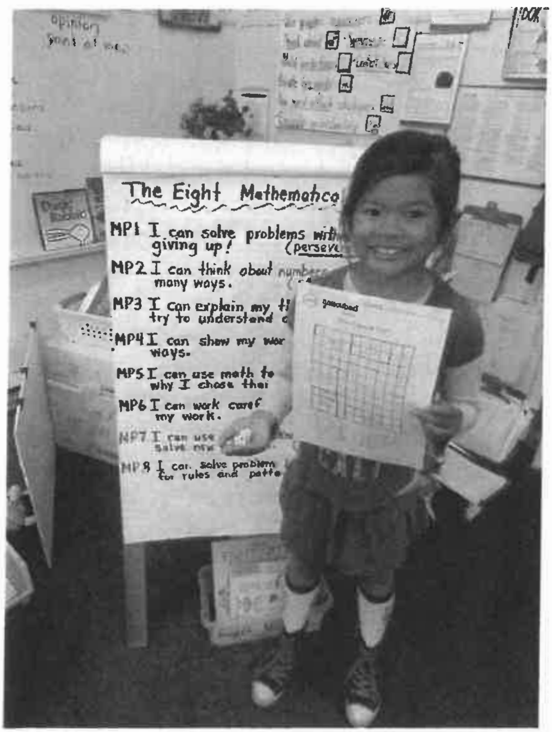


FIGURE 5.18 Third-grade student completes "How close to 100?"

The following are some of the students' important words as they reflected on the game:

"It challenged me to make my brain think."

"It was a fun way to explore math and learn."

"It gave me a lot of practice with multiplication."

"It's a fun way to learn multiplication facts."

"I learned that multiplication and area are related."

"I know now how division, multiplication, and area are all related because I can see it!"

The level of students' excitement in playing the game was matched only by the power of the mathematics they learned. It is noteworthy that as much as the students enjoyed the game, their comments all talked about the mathematics they learned. The students engaged in visual and numerical thinking about multiplication, division, and area, learning math facts through enjoyment and deep engagement—a far cry from the memorization of times tables.

In all of these six cases of mathematics excitement, the mathematics task was central (and supported by important teaching). The next section will review the important design aspects of these six tasks that can be applied to all mathematics tasks, regardless of grade level. It is also important to note that in all six cases students were working with each other, sometimes thinking alone but often collaborating on ideas, in classrooms where they were given positive growth mindset messages. I will now turn to the ways we can build these important design elements into any mathematics task.

From Cases of Mathematics Excitement to the Design of Tasks

We are emerging from an unproductive period in education. Since the No Child Left Behind Act was introduced by the Bush administration, teachers started to come under pressure to use "scripted" curriculum and pacing guides, even though they knew they were damaging their students. Many teachers felt deprofessionalized by this; they felt that important teaching decisions had been taken out of their hands. Fortunately, this time is ending; we are entering a much more positive time, with teachers being trusted to make important professional decisions. One of the aspects of teaching for a mathematics mindset that I am most excited about is the transformations that we can make in mathematics classrooms through giving important messages and opening up mathematics tasks. This opening up of tasks gives students the space to learn and is absolutely essential in developing mathematical mindsets.

A range of rich, open tasks are now available to teachers through websites, which I will list at the end of this chapter. But many teachers do not have time to search through websites.

Fortunately, teachers don't need to find new curriculum materials, as they can make adaptations to the tasks in the curriculum they use, opening them to create new and better opportunities for students. To do this, teachers may need to develop their own new mindsets as designers—that is, as people who can introduce a new idea and create new, enhanced learning experiences. The mathematical excitement I described earlier came, in a number of cases, from adapting a familiar task. In the growing shapes task, for example, the simple instruction for students to visualize the shapes growing changed everything, giving students access to understandings that would not have been possible otherwise. When teachers are designers, creating and adapting tasks, they are the most powerful teachers they can be. Any teacher can do this; it does not require special training. It involves knowing about the qualities of positive math tasks and approaching tasks with the mindset to improve them.

In designing and adapting math tasks for better learning, there are six questions that, if asked and acted upon in the task, increase their power incredibly. Some tasks are more suited to some questions than others, and many are naturally combined, but I am confident in saying every task will be made richer by paying attention to at least one of the following six questions.

1. Can You Open the Task to Encourage Multiple Methods, Pathways, and Representations?

There is nothing more important that teachers can do with tasks than to open them up so students are encouraged to think about different methods, pathways, and representations. When we open a task we transform its learning potential. Opening can happen in many ways. Adding a visual requirement, such as those shown in the growing shapes and negative space tasks, is a great strategy. Another way to open a task that is extremely mathematically productive is to ask students to make sense of their solutions.

Cathy Humphreys is wonderful teacher. In a book we coauthored, we show six video cases of Cathy teaching her seventh-grade class, accompanied by her lesson plans. One of the videos shows Cathy asking the students to solve: 1 divided by 2/3. This could be a closed, fixed mindset question with one right answer and one method, but Cathy transforms the task by adding two requirements: that students make sense of their solution and that they offer a visual proof (see Figure 5.19). She starts the lesson by saying "You may know a rule for solving this question, but the rule doesn't matter today, I want you to make sense of your answer, to explain why your solution *makes sense*."

In the video case we see that some students thought the answer was 6, because you can manipulate the set of numbers (1, 2, and 3), with no mathematical sense making, and make 6. But they struggled to show this visually or make sense of it. Others were able to show, in a range of different visual representations, why there were one and a half 2/3's "inside 1." The requirement for students to show their thinking visually and make sense of their answers transformed the question from a fixed to a growth mindset task, and created a wonderful lesson, filled with sense making and understanding.

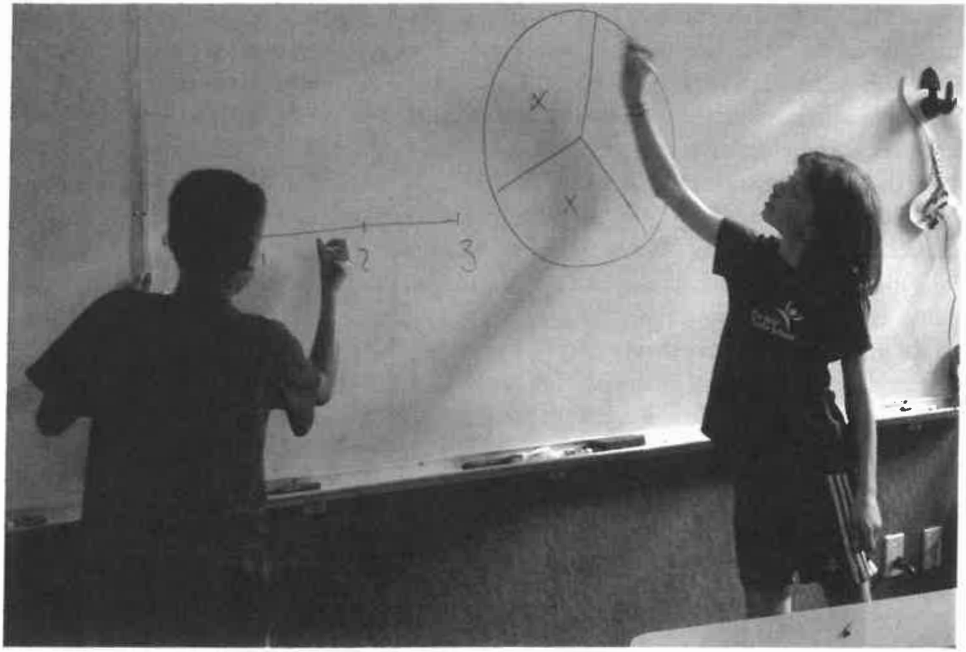


FIGURE 5.19 Students share solutions to 1 ÷ 2/3

2. Can You Make It an Inquiry Task?

When students think their role is not to reproduce a method but to come up with an idea, everything changes (Duckworth, 1991). The same mathematics content can be taught with questions that ask for a procedure or as questions that ask for students to think about ideas and use a procedure. For example, instead of asking students to find the area of a 12 by 4 rectangle, ask them how many rectangles they can find with an area of 24. This small adaptation changes students' motivation and understanding. In the inquiry version of the task, students use the formula for the area of a rectangle, but they also need to think about spatial dimensions and relationships, and what happens when one dimension changes (see Figure 5.20). The mathematics is more complex and exciting because students are using their ideas and thoughts.

Instead of asking students to name quadrilaterals with different qualities, ask them to come up with their own, as shown in Exhibit 5.5.

Another excellent task is the four 4's (see Exhibit 5.6). In this task you ask students to make all the numbers between 1 and 20 using four 4's and any operation; for example:

$$\sqrt{4} + \sqrt{4} + 4/4 = 5$$

This is a great activity for practicing operations, but it does not look like a practicing operations task, because the operations are beautifully embedded inside an inquiry task. When we posted

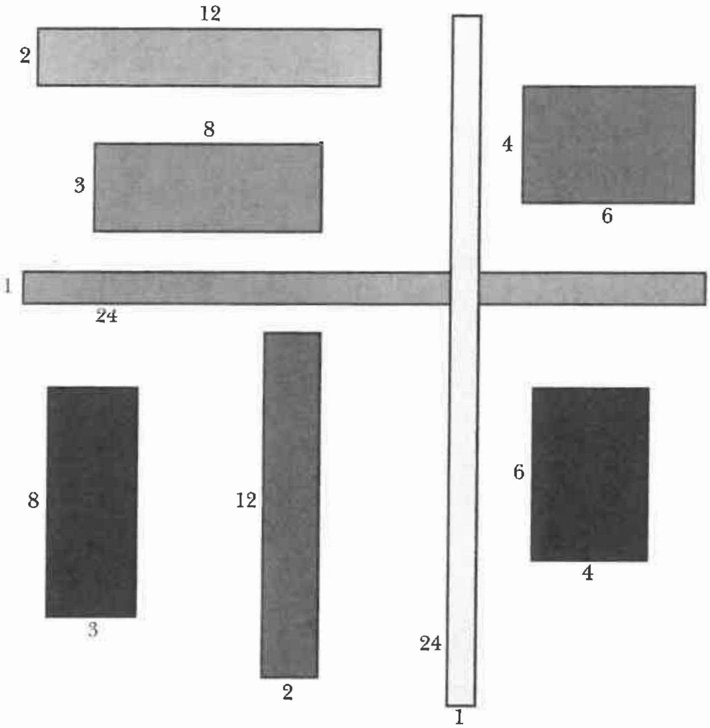


FIGURE 5.20 Rectangles with an area of 24

Find Quadrilaterals!

Pairs of Equal Sides \ Pairs of Parallel Sides	0	1	2
0			
1			
2			

Exhibit 5.5

Four 4's

Can you find every number between 1 and 20 using only four 4's and any operation?

Going beyond ...

Can you find more than one way to make each number with four 4's?

Can you go beyond 20?

Can you use four 4's to find negative integers?

Exhibit 5.6

this task on Youcubed.org, teachers told us the task was incredible. Here are two comments from Youcubed teachers:

> "My students were so inspired &excited with the four 4's they decided to investigate three 3's, and the sky was the limit."

> "The fours problem was amazing! I used it in my sixth-grade math class, and students were creating equations that led to discussions about distributive property, order of operations, variables ... it was fantastic!"

(The full task on Youcubed includes advice on ways to introduce the task and organize students; see https://www.youcubed.org/wim-day-1/.)

Another way to open a task and make it an inquiry task is to ask students to write a magazine article, a newsletter, or a short book about it. This structure can work with any content. At Railside, in ninth grade the students were asked to write a book on $y = mx + b$; they filled pages showing what this meant, how it could look visually, situations in which it could be used, and their ideas on the meaning of the equation. In a high school geometry unit that three of my graduate students at Stanford (Dan Meyer, Sarah Kate Selling and Kathy Sun) created with me, we asked students to write a newsletter on similarity, using photos, tasks, cartoons, and any other media they wanted to show what they knew about the topic (see https://www.youcubed.org/wp-content/uploads/The-Sunblocker1.pdf). Exhibit 5.7 is a general form of the newsletter assignment we gave out.

3. Can You Ask the Problem Before Teaching the Method?

When we pose problems for which students need to know a method before we introduce the method, we offer a great opportunity for learning and for using intuition. The tasks described earlier that exemplified this were the finding the largest enclosure area for a fence task and the finding the volume of a lemon task. But this design component can be used with any area of mathematics—in particular, for any teaching of a standard method or formula, such as the area of shapes, the teaching of pi, and statistical formulas such as mean, mode, range, and standard deviation. Exhibit 5.8 shows an example.

After students have worked out their own ways of finding averages and discussed them as groups and as a class, they could be taught the formal methods of mean, mode, and range.

4. Can You Add a Visual Component?

Visual understanding is incredibly powerful for students, adding a whole new level of understanding, as we saw in the growing shapes task. This can be provided through diagrams but also through physical objects, such as multilink cubes and algebra tiles. I spent my early years growing up with Cuisenaire rods, as my mother was training to be an elementary teacher. I spent many happy hours playing with the rods, ordering them and investigating mathematical patterns. In an online course designed to give students important mathematics strategies, I teach students to draw any mathematics problem or idea (see https://class.stanford.edu/courses/Education/EDUC115-S/Spring2014/about). Drawing is a powerful tool for mathematicians and mathematical problem solvers, most of whom draw any problem they are given. When students are stuck in math class, I often ask them to draw the problem out.

Newsletter

You are writing a newsletter to share your learning on this mathematics topic with your family and friends. You'll have the chance to describe your understanding of the ideas and write about why the mathematical ideas you have learned are important. You'll also describe a couple of activities that you worked on that were interesting to you.

In creating your newsletter, you can draw on the following resources:

- Photos of different activities
- Sketches
- Cartoons
- Interviews/surveys

To refresh your memory, here are some of the activities we've worked on:

Please prepare the following four sections. You can change the titles of the sections to fit your work.

Headline News Explain the big idea of the mathematics and what it means in at least two different ways. Use words, diagrams, pictures, numbers, and equations.	New Discoveries Choose at least two different activities from the work we have done that helped you understand the concepts. For each activity: • Explain why you chose the activity. • Explain what you learned about through the activity. • Explain what was challenging about the activity. • Explain the strategies you used to address your challenge.
Connections Choose one additional activity that helped you learn a mathematical idea or process that you can connect to some other learning. • Explain why you chose the activity. • Explain the big mathematical idea you learned from the activity. • Explain what you connected this idea to and how you see the connection. • Explain the importance of the connection and how you might use this in the future.	The Future Write a summary for the newsletter that addresses the following: • What is the big mathematical idea useful for? • What questions do you still have about the big idea?

Exhibit 5.7

The Long Jump

You are going to try out for the long jump team, for which you need an average jump of 5.2 meters. The coach says she will look at your best jump each day of the week and average them out. These are the five jumps you recorded that week:

	Meters
Monday	5.2
Tuesday	5.2
Wednesday	5.3
Thursday	5.4
Friday	4.4

Unfortunately, Friday's was a low score because you weren't feeling that well!

How could you work out an average that you think would fairly represent your jumping? Work out some averages in different ways and see which you think is most fair, then give an argument for why you think it is fairest. Explain your method and try and convince someone that your approach is best.

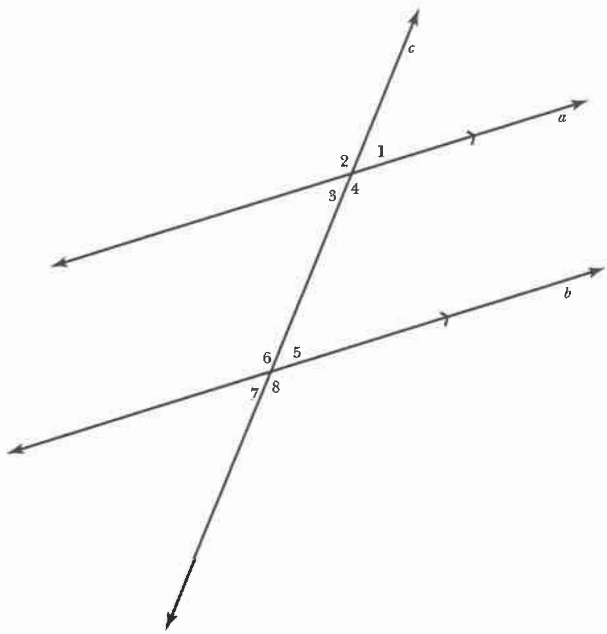
Exhibit 5.8

Railside School, the highly successful school I studied, asked students to show connections through color coding. For example, when teaching algebra, they asked students to show functional relationships in many forms: as an expression, as a picture, in words, and on a graph. Many schools ask for these different representations. Railside was unusual in that they asked the students to show relationships in color; for example, to show the x in the same color in an expression, on the graph, and in the diagram. Chapter Seven, which describes the Railside approach in more detail, shows one of their color coding tasks. In other topic areas—for example, when asking students to identify congruent, vertical, and supplementary angles—you could also ask them to color and write about as many relationships as they can, using color to highlight the relationships. Exhibit 5.9 and Figure 5.21 show an example.

Further examples of color coding are given in Chapter Nine.

Parallel Lines and a Transversal

1. Use color coding to identify congruent angles.
2. Identify vertical and supplementary angles.
3. Write about the relationships you see. Use the color from your diagram in your writing.



Vertical Angles:

Supplementary Angles:

Relationships:

Exhibit 5.9

5. Can You Make It Low Floor and High Ceiling?

All of the preceding problems are low floor and high ceiling. The breadth of the space inside them means that they are accessible to a wide range of students and they extend to high levels.

One way to make the floor lower is to always ask students how they see a problem. This is an excellent question for other reasons too, as I have explained.

A great strategy for making a task higher ceiling is to ask students who have finished a question to write a new question that is similar but more difficult. When we were teaching a group of

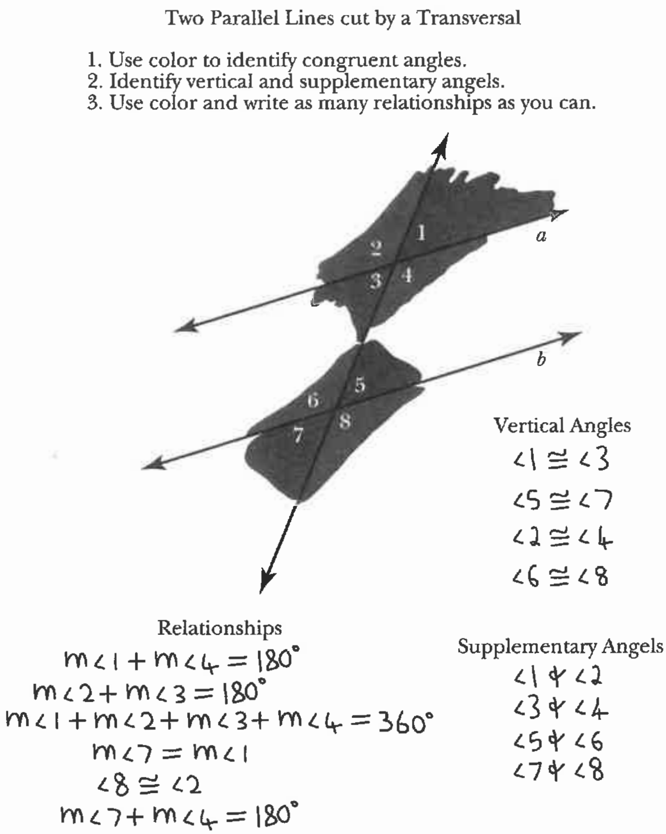


FIGURE 5.21 Color coding angles

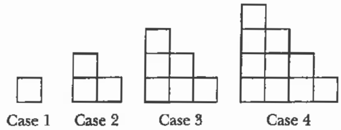
heterogeneous students in summer school, we used this strategy a lot to great effect. For example, when one student, Alonzo, finished the staircase task, which asked students to think about pattern growth and the nth case (see Exhibit 5.10), he asked a harder question. He asked how a staircase extending in four directions would grow and the number of cubes in the nth case (see Figure 5.22).

When students are invited to ask a harder question, they often light up, totally engaged by the opportunity to use their own thinking and creativity. This is an easy extension for teachers to use and one that they can have available in any lesson. With any set of mathematics questions, consider giving students a task like this:

> "Now you write a question; try to make it hard ☺"

Students can give their questions to other students, who can be encouraged to write questions for each other. This is a particularly good strategy to use for students who work faster than other students or who complain that work is too easy for them, as it involves deep and difficult thinking.

Staircase



How do you see the pattern growing?

How many would be in the 100th case?

What about the nth case?

Exhibit 5.10

6. Can You Add the Requirement to Convince and Reason?

Reasoning is at the heart of mathematics. When students offer reasons and critique the reasoning of others, they are being inherently mathematical and preparing for the high-tech world they will be working in, as well as the Common Core. Reasoning also gives students access to understanding. In my four-year study of different schools, we found that reasoning had a particular role to play in the promotion of equity, as it helped to reduce the gap between students who understood and students who were struggling. In every math conversation, students were asked to reason, explaining why they had chosen particular methods and why they made sense. This opened up mathematical pathways and allowed students who had not understood to both gain understanding and ask questions, adding to the understanding of the original student.

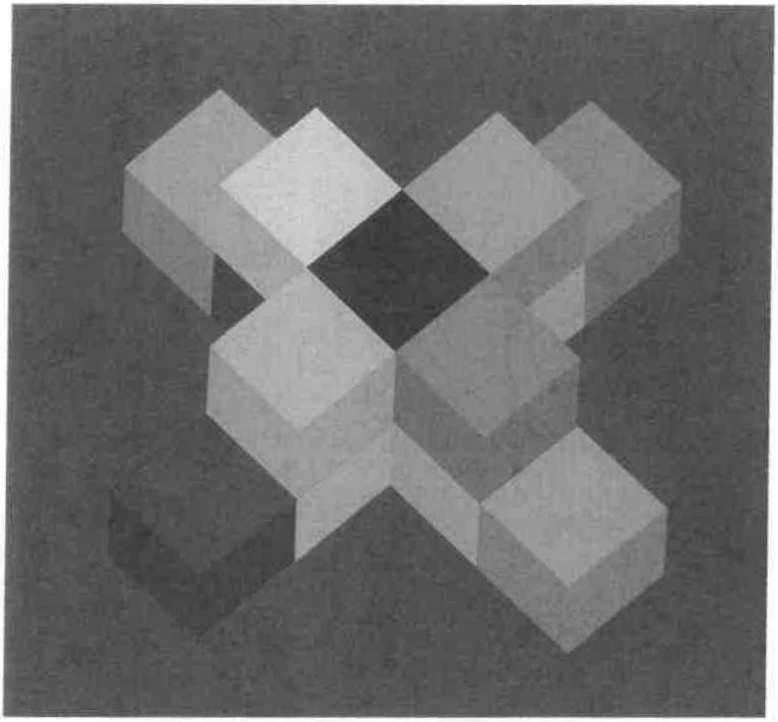


FIGURE 5.22 Alonzo's extension problem

I like to accompany one of my favorite tasks for encouraging reasoning with a pedagogical strategy that has many benefits. I learned this strategy from Cathy Humphreys, who asks her students to be skeptics. She explains that there are three levels of being convincing (Boaler & Humphreys, 2005):

Convince yourself

Convince a friend

Convince a skeptic

It is fairly easy to convince yourself or a friend, but you need high levels of reasoning to convince a skeptic. Cathy tells her students that they need to be skeptics, pushing other students to always give full and convincing reasons.

A perfect task to teach and encourage higher levels of reasoning that can be accompanied by the skeptic role was developed by Mark Driscoll; it is called "paper folding." I have used this task with a range of different groups, always with very high levels of engagement. Teachers tell me they love this task, as it often lets students shine who don't typically get that opportunity. In this task, students work in pairs with a square piece of paper. They are asked to fold the paper to make new shapes. Exhibit 5.11 shows the five, progressively more challenging questions (see Figure 5.23).

Paper Folding

Work with a partner. Take turns being the skeptic or the convincer. When you are the convincer, your job is to be convincing! Give reasons for all of your statements. Skeptics must be skeptical! Don't be easily convinced. Require reasons and justifications that make sense to you.

For each of the following problems, one person should make the shape and then be convincing. Your partner is the skeptic. When you move to the next question, switch roles.

Start with a square sheet of paper and make folds to construct a new shape. Then, explain how you know the shape you constructed has the specified area.

1. Construct a square with exactly 1/4 the area of the original square. Convince your partner that it is a square and has 1/4 of the area.

2. Construct a triangle with exactly 1/4 the area of the original square. Convince your partner that it has 1/4 of the area.

3. Construct another triangle, also with 1/4 the area, that is not congruent to the first one you constructed. Convince your partner that it has 1/4 of the area.

4. Construct a square with exactly 1/2 the area of the original square. Convince your partner that it is a square and has 1/2 of the area.

5. Construct another square, also with 1/2 the area, that is oriented differently from the one you constructed in 4. Convince your partner that it has 1/2 of the area.

Source: Adapted from Driscoll, 2007, p.90, http://heinemann.com/products/E01148.aspx

Exhibit 5.11

When I have given this task to teachers they have struggled for a long time on question 5, some working well into the evening after a full day of professional development, enjoying every moment. Their engagement is enhanced with having a physical shape to consider and change, but also by the need to be convincing. When I give students and teachers this task, I ask for the pairs to take turns, with one folding and convincing and one being the skeptic; then they switch for the

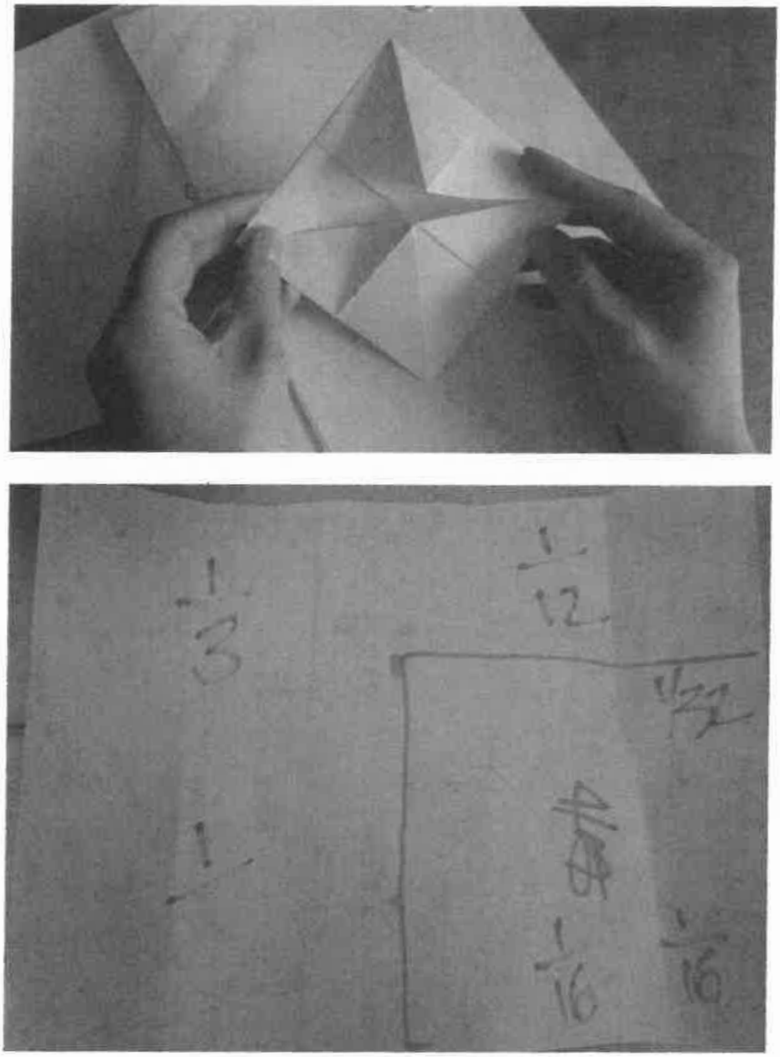
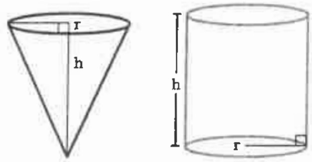


FIGURE 5.23 Teachers work on paper folding task

next question. When I ask students to play the role of being the skeptic, I explain that they need to demand to be fully convinced. Students really enjoy challenging each other for convincing reasons, and this helps them learn mathematical reasoning and proof. As a teacher you may want to model what a fully convincing answer is, by asking students follow-up questions if they have not been convincing enough.

Another example of a task that involves convincing is shown in Exhibit 5.12. The request for students to reason and be convincing can be applied to any mathematics problem or task.

Cone and Cylinder



The height and radius of the cone and cylinder are the same. What is the relationship between the volume of the cone and the volume of the cylinder? Make a conjecture and try to convince other students. Use drawings, models, and color coding to be convincing.

Exhibit 5.12

Conclusion

When mathematics tasks are opened for different ways of seeing, different methods and pathways, and different representations, everything changes.

Questions can move from being closed, fixed mindset math tasks to growth mindset math tasks, with space within them to learn. To summarize, these are my five suggestions that can work to open mathematics tasks and increase their potential for learning:

1. Open up the task so that there are multiple methods, pathways, and representations.
2. Include inquiry opportunities.
3. Ask the problem before teaching the method.
4. Add a visual component and ask students how they see the mathematics.
5. Extend the task to make it lower floor and higher ceiling.
6. Ask students to convince and reason; be skeptical.

Further examples of tasks with these design features are given in Chapter Nine.

If you take the opportunity to modify tasks in these ways, you will be offering your students more and deeper learning opportunities. I have really enjoyed all the times I have seen students working on rich open mathematics tasks, and I have taught with them myself, as students are so excited by them. They love to make connections, which are so important in mathematics, and visual, creative mathematics is inspiring to students. A week of mathematics lessons that include the design features discussed in this chapter, and that are appropriate for grades 3 to 9, can be viewed and downloaded freely here: https://www.youcubed.org/week-of-inspirational-math/.

When I trialed these lessons in middle school classrooms, I experienced parents rushing up to me to tell me that these lessons had changed mathematics for their children. Some parents told me that their children had always disliked math until they took these lessons and saw mathematics in a completely different light. With a design and mathematical mindset, teachers (and parents) can create and transform mathematics tasks, giving all students the rich mathematics environment that they deserve. We cannot wait for publishing companies to realize these changes are needed and make the necessary changes, but teachers can make these changes–creating open, engaging mathematics environments for all of their students.

The following websites provide mathematics tasks that incorporate one or more of the features I have highlighted:

- Youcubed: www.youcubed.org
- NCTM: www.nctm.org (membership required to access some of the resources)
- NCTM Illuminations: http://illuminations.ntcm.org
- Balanced Assessment: http://balancedassessment.concord.org
- Math Forum: www.mathforum.org
- Shell Center: http://map.mathshell.org/materials/index.php
- Dan Meyer's resources: http://blog.mrmeyer.com/
- Geogebra: http://geogebra.org/cms/
- Video Mosaic project: http://videomosaic.org/
- NRich: http://nrich.maths.org/
- Estimation 180: http://www.estimation180.com
- Visual Patterns; grades K–12: http://www.visualpatterns.org
- Number Strings: http://numberstrings.com
- Mathalicious, grades 6–12; real-world lessons for middle and high school: http://www.mathalicious.com